GROMOV–WITTEN THEORY OF $\mathbb{C}P^1$ AND INTEGRABLE HIERARCHIES

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Abstract. The ancestor Gromov–Witten invariants of a compact Kähler manifold X can be organized in a generating function called the total ancestor potential of X. In this paper, we construct Hirota Quadratic Equations (HQE shortly) for the total ancestor potential of $\mathbb{C}P^1$. The idea is to adopt the formalism developed in [G1, GM] to the mirror model of $\mathbb{C}P^1$. We hope that the ideas presented here can be generalized to other manifolds as well.

As a corollary, using the twisted loop group formalism from [G3], we obtain a new proof of the following version of the Toda conjecture: the total descendant potential of $\mathbb{C}P^1$ (known also as the partition function of the $\mathbb{C}P^1$ topological sigma model) is a tau-function of the Extended Toda Hierarchy.

1. Introduction

Let X be a compact Kähler manifold. Denote $X_{g,m,d}$ the moduli space of degree $d \in H_2(X,\mathbb{Z})$ *stable maps* to X of genus-g m-pointed complex curves. In case $X = pt$ the moduli space is denoted by $\overline{\mathcal{M}}_{g,m}$. The *total ancestor* potential of X is defined as the following generating function of Gromov–Witten invariants of X:

$$\mathcal{A}_\tau = \exp\left(\sum_{g\geq 0} \epsilon^{2g-2}\overline{\mathcal{F}}_\tau^{(g)}\right),$$

where $\tau \in H^*(X;\mathbb{C})$ is a parameter and $\overline{\mathcal{F}}_\tau^{(g)}$ are the genus-g ancestor potentials:

$$\overline{\mathcal{F}}_\tau^{(g)} = \sum_{l,m,d} \frac{Q^d}{l!m!}\int_{[X_{g,l+m,d}]} \wedge_{i=1}^m \left(\sum_{k\geq 0} \mathrm{ev}_i^*\, t_k \overline{\psi}_i^k\right) \wedge_{i=m+1}^{m+l} \mathrm{ev}_i^*\,\tau,$$

$[X_{g,m,d}]$ — the virtual fundamental class of $X_{g,m,d}$,
$t_n \in H := H^*(X, \mathbb{C}[[Q]])$ — arbitrary cohomology classes of X,
$\mathrm{ev}_j : X_{g,m,d} \to X$ — evaluation at the j-th marked point,
Q^d — the elements of the Novikov ring $\mathbb{C}[[Q]] := \mathbb{C}[[$ Mori cone of $X]]$,
$\overline{\psi}_i = \pi^*(\psi_i)$, $i = 1, ..., m$, where the map $\pi : X_{g,m+l,d} \to \overline{\mathcal{M}}_{g,m+l} \to \overline{\mathcal{M}}_{g,m}$ is the composition of the contraction map with the operation of forgetting all markings except the first m ones and ψ_i are the 1-st Chern classes of the universal cotangent lines over the moduli space of curves $\overline{\mathcal{M}}_{g,m}$.
It is a formal function in the sequence of vector variables $t_0, t_1, t_2, ...$ and τ.

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Let $\mathcal{H} = H((z^{-1}))$ be the space of formal Laurent series in z^{-1}, equipped with the following symplectic structure:

$$\Omega(f,g) := \frac{1}{2\pi i}\oint (f(-z),g(z))\ dz, \tag{1.1}$$

where $(\ ,\ )$ is the Poincaré pairing on H. The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, defined by the Lagrangian subspaces $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$, identifies $\mathcal{H}$ with the cotangent bundle $T^*\mathcal{H}_+$.

Denote $\mathcal{B}$ *the Bosonick Fock space* which consists of functions on $\mathcal{H}_+$ which belong to the formal neighborhood of $-\mathbf{1}z$ ($\mathbf{1}$ is the unity in H) i.e., if we let $\mathbf{q}(z) = \sum_{k\geq 0} q_k z^k \in \mathcal{H}_+$ then $\mathcal{B}$ is the space of formal functions in the sequence of vector variables $q_0, q_1+\mathbf{1}, q_2, \dots$. The total ancestor potential $\mathcal{A}_\tau$ is identified with a vector in $\mathcal{B}$ via *the dilaton shift* $\mathbf{t}(z) = \mathbf{q}(z) + z$, where $\mathbf{t}(z) = \sum t_k z^k$.

Any $\mathbf{f} \in \mathcal{H}$ can be written uniquely as

$$\mathbf{f} = \sum_{k\geq 0,a} q_{k,\alpha}\ \phi_a z^k + p_{k,a}\ \phi^a(-z)^{-1-k},$$

where $\{\phi_a\}$ is a basis of H and $\{\phi^a\}$ is its dual with respect to the Poincaré pairing. The coefficients $p_{k,a}$ $q_{k,a}$ are coordinate functions on $\mathcal{H}$ which form a Darboux coordinate system. Thus the formulas

$$\hat{q}_{k,a} := q_{k,a}/\epsilon, \quad \hat{p}_{k,a} := \epsilon\partial/\partial q_{k,a} \tag{1.2}$$

define a representation of the Heisenberg Lie algebra generated by the linear Hamiltonians on the Fock space $\mathcal{B}$. Given a vector $\mathbf{f} \in \mathcal{H}$ we define *a vertex operator* acting on $\mathcal{B}$: $e^{\widehat{\mathbf{f}}} := (e^{\mathbf{f}})\hat{}:= e^{\hat{\mathbf{f}}_-}e^{\hat{\mathbf{f}}_+}$, where $\mathbf{f}_\pm$ is the projection of $\mathbf{f}$ on $\mathcal{H}_\pm$ and $\mathbf{f}_\pm$ is identified with the linear Hamiltonian $\Omega(\cdot, \mathbf{f}_\pm)$.

The far reaching goal is to construct HQE for the total ancestor potential of X in terms of vertex operators acting on the Fock space $\mathcal{B}$. In this paper we will solve this problem for $X = \mathbb{C}P^1$. The vertex operators will be defined through the mirror model of $\mathbb{C}P^1$ and it is not hard to generalize the definition to other manifolds as well. However, it is hard to generalize the HQE.

When $X =$pt, the total ancestor potential is in fact independent of the parameter $\tau \in H^*(\mathrm{pt}, \mathbb{C}) \cong \mathbb{C}$ (see subsection 3.1) and it will be denoted by $\mathcal{A}_{\mathrm{pt}}$. By definition, $\mathcal{A}_{\mathrm{pt}}$ is the total ancestor potential of A_1 singularity. Thus, according to the Corollary of Proposition 2 in [G1], $\mathcal{A}_{\mathrm{pt}}$ satisfies a family of HQE parametrized by $\tau \in \mathbb{C}$. Let us write down this family of HQE explicitly. We will take a slightly different point of view from [G1] which will be used also for the case $X = \mathbb{C}P^1$.

Let $f_\tau : \mathbb{C} \to \mathbb{C}$ be the function $f_\tau(x) = x^2/2 + \tau$. For fixed τ, f_τ is a Morse function with critical point $x_1 = 0$ and critical value $u_1 = \tau$. Pick a reference point $\lambda_0 \in \mathbb{C}\backslash\{u_1\}$ and a path C_1 connecting λ_0 with u_1 and such that it approaches u_1 along a straight segment. Denote by β the corresponding *Lefschetz thimble* i.e., the relative homology cycle in $H_1(\mathbb{C}, f_\tau^{-1}(\lambda_0); \mathbb{Z})$ represented by $f_\tau^{-1}(C_1)$ with an

arbitrary choice of the orientation. For each $n \in \mathbb{Z}$ we define a period vector $I^{(n)}_\beta$:

$$
\begin{aligned}
I^{(-k)}_\beta(\lambda,\tau) &= (\partial/\partial\lambda)\int_{\beta(\lambda)} \frac{(\lambda - f_\tau(x))^k}{k!}\omega, \\
I^{k}_\beta(\lambda,\tau) &= \partial^k_\lambda I^{(0)}_\beta(\lambda,\tau),
\end{aligned}
$$

where k is a non-negative integer, $\omega = dx$, and $\beta(\lambda) \in H_1(\mathbb{C}, f_\tau^{-1}(\lambda);\mathbb{Z})$ is a Lefschetz thimble obtained by extending β along a path C connecting λ_0 and λ. The period vectors are multivalued functions on $\mathbb{C}\backslash\{u_1\}$ with values in H.

If $\alpha = r\beta$ for some $r \in \mathbb{Q}$, then we define:

$$
I^{(n)}_\alpha = rI^{(n)}_\beta, \quad \mathbf{f}^\alpha_\tau(\lambda) = \sum_n I^{(n)}_\alpha(\lambda,\tau)(-z)^n, \quad \Gamma^\alpha_\tau = e^{\widehat{\mathbf{f}}^\alpha_\tau}.
$$

Let $\alpha = \beta/2$. Then the integrals defining the corresponding period vectors can be computed explicitly and we get:

$$
\Gamma^\alpha_\tau = \exp\left(\pm\sum_{n\geq 0}\frac{(2(\lambda-\tau))^{n+1/2}}{(2n+1)!!}\frac{q_n}{\epsilon}\right)\exp\left(\mp\sum_{n\geq 0}\frac{(2n-1)!!}{(2(\lambda-\tau))^{n+1/2}}\epsilon\partial_{q_n}\right), \tag{1.3}
$$

where the sign depends on the choice of the path C. We remark that

$$
\frac{(2n-1)!!}{(2(\lambda-\tau))^{1/2+n}} = \left(-\frac{d}{d\lambda}\right)^n \frac{1}{\sqrt{2(\lambda-\tau)}}, \quad \frac{(2(\lambda-\tau))^{1/2+n}}{(2n+1)!!} = \left(\frac{d}{d\lambda}\right)^{-1-n}\frac{1}{\sqrt{2(\lambda-\tau)}}.
$$

Let

$$
c_\alpha(\lambda,\tau) := \lim_{\epsilon\to 0}\ \exp\left(\int_\lambda^{u_1+\epsilon}\left(I^{(0)}_\alpha(\xi,\tau), I^{(0)}_\alpha(\xi,\tau)\right)d\xi - \langle\alpha,\alpha\rangle\int_1^\epsilon\frac{d\xi}{\xi}\right) = \frac{1}{\sqrt{\lambda-\tau}},
$$

where $\langle\ ,\ \rangle$ is the intersection pairing in $H_1(\mathbb{C}, f_\tau^{-1}(\lambda);\mathbb{Z}) : \langle\beta,\beta\rangle = 2$. Here, the limit $\epsilon \to 0$ is taken along a straight segment such that $u_1+\epsilon$ parametrizes the end of the path C_1. The integration path in the 1-st integral is $C_1(\epsilon)\circ C^{-1}$, where $C_1(\epsilon)$ is the path obtained from C_1 by truncating the line segment between $u_1+\epsilon$ and u_1, and in the 2-nd one — a straight segment between 1 and ϵ.

The HQE for the ancestor potential of a point can be stated this way:

$$
c_\alpha\left(\Gamma^\alpha_\tau\otimes\Gamma^{-\alpha}_\tau - \Gamma^{-\alpha}_\tau\otimes\Gamma^\alpha_\tau\right)(\mathcal{A}_{\mathrm{pt}}\otimes\mathcal{A}_{\mathrm{pt}})\,d\lambda \quad \text{is regular in} \quad \lambda. \tag{1.4}
$$

Here $\mathcal{A}_{\mathrm{pt}}\otimes A_{\mathrm{pt}}$ means the function $A_{\mathrm{pt}}(\mathbf{q}')A_{\mathrm{pt}}(\mathbf{q}'')$ of the two copies of the variable $\mathbf{q} = (q_0, q_1, ...)$, and the vertex operators in $\Gamma^{\pm\alpha}_\tau\otimes\Gamma^{\mp\alpha}_\tau$ preceding (respectively — following) $\otimes$ act on $\mathbf{q}'$ (respectively — on $\mathbf{q}''$). The expression in (1.4) is in fact single-valued for λ near ∞. Passing to the variables $\mathbf{x} = (\mathbf{q}'+\mathbf{q}'')/2$ and $\mathbf{y} = (\mathbf{q}'-\mathbf{q}'')/2$ and using Taylor's formula one can expand (1.4) into a power series in $\mathbf{y}$ with coefficients which are Laurent series in λ^{-1} (whose coefficients are polynomials in $\mathcal{A}_{\mathrm{pt}}$ and its partial derivatives). The regularity condition in (1.4) means, by definition, that all the Laurent series in λ^{-1} are polynomials in λ.

The HQE (1.4) for $\mathcal{A}_{\mathrm{pt}}$ are consequence of Witten's conjecture [W], proved by Kontsevich [Ko], and the string equation.

Let $X = \mathbb{C}P^1$ and $\phi_0 = \mathbf{1}$, $\phi_1 = P$ (P is the cohomology class of the hyperplane section of $\mathbb{C}P^1$) be a basis in $H = H^*(\mathbb{C}P^1;\mathbb{C})$. We will assume that the parameter of the ancestor potential is $\tau = tP$. Let $f_t : \mathbb{C}^* \to \mathbb{C}$ be the function $f_t(x) = x + (Qe^t/x)$. For fixed t, f_t is a Morse function with critical points $x_{1/2} = \pm\sqrt{Q}e^{t/2}$ and critical values $u_{1/2} = \pm 2\sqrt{Q}e^{t/2}$. Pick a reference point $\lambda_0 \in \mathbb{C}\backslash\{u_1, u_2\}$ and paths $C_i,\ i = 1, 2$ connecting λ_0 with u_i and such that C_i approaches u_i along a straight segments. Denote β_i the corresponding *Lefschetz thimbles* i.e., the relative homology cycles in $H_1(\mathbb{C}^*, f_t^{-1}(\lambda_0);\mathbb{Z})$ represented by $f_t^{-1}(C_i)$. We fix the orientation on β_i in such a way that $\phi := \beta_1 + \beta_2$ is a cycle homologous to the circle (with the counter-clockwise orientation) around the puncture of the punctured plane $\mathbb{C}^*$. For each $n \in \mathbb{Z}$ we define period vectors $I^{(n)}_{\beta_i}$:

$$
\begin{aligned}
(I^{(-k)}_{\beta_i}(\lambda, t), 1) &= (\partial/\partial\lambda) \int_{\beta_i(\lambda)} \frac{(\lambda - f_t(x))^k}{k!}\omega, \\
(I^{(-k)}_{\beta_i}(\lambda, t), P) &= -(\partial/\partial t) \int_{\beta_i(\lambda)} \frac{(\lambda - f_t(x))^k}{k!}\omega, \\
I^{k}_{\beta_i}(\lambda, t) &= \partial^k_\lambda I^{(0)}_{\beta_i}(\lambda, t),
\end{aligned}
$$

where k is a non-negative integer, $\omega = dx/x$, and $\beta_i(\lambda) \in H_1(\mathbb{C}^*, f_t^{-1}(\lambda);\mathbb{Z})$ is a Lefschetz thimble obtained by extending β_i along a path C connecting λ_0 and λ. The period vectors are multivalued functions on $\mathbb{C}\backslash\{u_1, u_2\}$ with values in H.

If $\alpha = r_1\beta_1 + r_2\beta_2$ for some $r_1, r_2 \in \mathbb{Q}$, then we define:

$$
I^{(n)}_\alpha = r_1 I^{(n)}_{\beta_1} + r_2 I^{(n)}_{\beta_2}, \quad \mathbf{f}^\alpha_\tau(\lambda) = \sum_n I^{(n)}_\alpha(\lambda, t)(-z)^n, \quad \Gamma^\alpha_\tau = e^{\widehat{\mathbf{f}}^\alpha_\tau}.
$$

Let $\alpha = \beta_1/2$. Introduce the function

$$
c_\alpha(\lambda, \tau) = \lim_{\epsilon \to 0} \exp\left(\int_\lambda^{u_1+\epsilon} \left(I^{(0)}_\alpha(\xi, \tau), I^{(0)}_\alpha(\xi, \tau) \right) d\xi - \langle \alpha, \alpha \rangle \int_1^\epsilon \frac{d\xi}{\xi} \right),
$$

where $\langle\ ,\ \rangle$ is the intersection pairing in $H_1(\mathbb{C}^*, f_\tau^{-1}(\lambda);\mathbb{Z}) : \langle\beta_1, \beta_1\rangle = \langle\beta_2, \beta_2\rangle = 2$ and $\langle\beta_1, \beta_2\rangle = -2$. The limit $\epsilon \to 0$ is taken along a straight segment such that $u_1 + \epsilon$ parametrizes the end of the path C_1. The 1-st integration path is the path $C_1(\epsilon) \circ C^{-1}$, where $C_1(\epsilon)$ is obtained from C_1 by truncating the line segment between $u_1 + \epsilon$ and u_1, and the 2-nd one — a straight segment between 1 and ϵ.

The expression similar to (1.4) is not single valued near ∞ – we will prove that the vertex operators $\Gamma^{\pm\alpha}_\tau \otimes \Gamma^{\mp\alpha}_\tau$ under the analytic continuation $\lambda = \infty$ are multiplied by the monodromy factors $\Gamma^{\pm\phi}_\tau \otimes \Gamma^{\mp\phi}_\tau$. To offset the complication we will generalize the concept of vertex operators.

We will allow vertex operators with coefficients in the algebra $\mathcal{A}$ of differential operators $\sum_{0\le k\le N} a_k(x;\epsilon)\partial^k_x$, where each a_k is a formal Laurent series in ϵ with coefficients smooth functions in x. We equip $\mathcal{A}$ with the *anti-involution* # which acts on the generators of $\mathcal{A}$ as follows:

$$
(\epsilon\partial_x)^\# = -\epsilon\partial_x + \log Q, \quad x^\# = x.
$$

Let w_τ and v_τ be the vectors in $\mathcal{H}$ defined by:

$$w_\tau = -Pz^{-1} - \sum_{k\geq 0\,,a=0,1} \langle P\psi^k, \phi_a\rangle_{0,2}(\tau)\phi^a z^{-k-2}, \tag{1.5}$$

$$v_\tau = \quad \mathbf{1} \quad + \sum_{k\geq 0\,,a=0,1} \langle \psi^k, \phi_a\rangle_{0,2}(\tau)\phi^a z^{-k-1}, \tag{1.6}$$

where the correlator notation stands for

$$\langle \phi_\alpha\psi^k, \phi_\beta\rangle_{0,2}(\tau) = \sum_{n,d} \frac{Q^d}{n!}\int_{[X_{0,n+2,d}]} \mathrm{ev}_1^*(\phi_\alpha)\psi_1^k \wedge \mathrm{ev}_2^*(\phi_\beta)\wedge (\wedge_{i=3}^{n+2}\mathrm{ev}_i^*(\tau))\ ,$$

where ψ_1 is the 1-st Chern class of the cotangent line (at the 1-st marked point) bundle over the moduli space $X_{0,n+2,d}$.

Introduce the vertex operator, acting on the algebra

$$\pi := \mathcal{B}\otimes_{\mathbb{C}((\epsilon))}\mathcal{A} = \left\{ \sum_{k=1}^N a_k(x,\mathbf{q};\epsilon)\partial_x^k \right\}$$

of differential operators with coefficients in the Fock space $\mathcal{B}$,

$$\Gamma^\delta_\tau = \exp\left((\hat{\mathbf{f}}_\tau^{\phi/2\pi i} - \hat{w}_\tau)(\epsilon\partial_x - \log\sqrt{Q})\right)\exp((x/\epsilon)\hat{v}_\tau).$$

We define the following HQE: *a function $\mathcal{T}\in\mathcal{B}$ satisfies the HQE of $\mathbb{C}P^1$ if*

$$\left(\Gamma^{\delta\#}_\tau\otimes\Gamma^\delta_\tau\right) c_\alpha\left(\Gamma^\alpha_\tau\otimes\Gamma^{-\alpha}_\tau - \Gamma^{-\alpha}_\tau\otimes\Gamma^\alpha_\tau\right)(\mathcal{T}\otimes\mathcal{T})\ d\lambda \tag{1.7}$$

computed at $\mathbf{q}'$ and $\mathbf{q}''$ such that $\hat{w}'_\tau - \hat{w}''_\tau = m$, is regular in λ for each $m\in\mathbb{Z}$.

The expression (1.7) is interpreted as taking values in the algebra $\pi\otimes_{\mathcal{A}}\pi$ of differential operators with coefficients depending on $\mathbf{q}', \mathbf{q}'', \epsilon$ and λ. We will show that for every $r\in\mathbb{Q}$ the following identity holds:

$$\left(\Gamma^{\delta\#}_\tau\otimes\Gamma^\delta_\tau\right)\left(\Gamma^{r\,\phi}_\tau\otimes\Gamma^{-r\,\phi}_\tau\right) = e^{2\pi i(\hat{w}_\tau\otimes 1 - 1\otimes\hat{w}_\tau)\,r}\,\Gamma^{\delta\#}_\tau\otimes\Gamma^\delta_\tau.$$

Thus when $\hat{w}'_\tau - \hat{w}''_\tau\in\mathbb{Z}$, the expression (1.7) is single-valued near $\lambda=\infty$. After the change $\mathbf{y} = (\mathbf{q}'-\mathbf{q}'')/2, \mathbf{x} = (\mathbf{q}'+\mathbf{q}'')/2$ and the substitution [1]

$$y_{0,0} = -\frac{m\epsilon}{2} + \sum_{k=0}^\infty\sum_{a=0,1}\langle P\psi^k,\phi_a\rangle_{0,2}(\tau)(-1)^{k+1}y_{k+1,a}$$

it expands (for each integer m) as a power series in $\mathbf{y}$ ($y_{0,0}$ excluded) with coefficients which are Laurent series in λ^{-1} (whose coefficients are *differential operators in x* depending on $\mathbf{x}$ via $\mathcal{T}$, its translations and partial derivatives).

Theorem 1.1. *The total ancestor potential of $\mathbb{C}P^1$ satisfies the HQE* (1.7).

[1] according to our quantization rules: $\hat{w}_\tau = -q_{0,0}/\epsilon + \sum_{k,a}(-1)^k\left\langle P\psi^k,\phi_a\right\rangle_{0,2}(\tau)q_{k+1,a}/\epsilon$

As an application of Theorem 1.1, we will give a proof of the so called Toda conjecture (see [EY, Ge1, OP2, CDZ, Z]) about Gromov – Witten invariants of $\mathbb{C}P^1$.

Let X be a compact Kähler manifold. The *total descendant potential* of X is defined as the following generating function for Gromov – Witten invariants of X:

$$\mathcal{D}_X := \exp \sum_{g\geq 0} \epsilon^{2g-2}\mathcal{F}^{(g)}, \quad \mathcal{F}^{(g)} := \sum_{m,d} \frac{Q^d}{m!} \int_{[X_{g,m,d}]} \prod_{j=1}^{m} \sum_{n\geq 0} \mathrm{ev}_j^*(t_n)\psi_j^n,$$

where ψ_j are the 1-st Chern classes of the universal cotangent lines over $X_{g,m,d}$ and $t_n \in H^*(X, \mathbb{Q}[[Q]])$ are arbitrary cohomology classes of X. It is identified with a vector in the Fock space $\mathcal{B}$ via the dilaton shift $\mathbf{t}(z) = \mathbf{q}(z) + z$.

Corollary 1.2. *The total descendant potential of* $\mathbb{C}P^1$ *is a tau-function of the Extended Toda Hierarchy.*

The Toda Conjecture was suggested by T. Eguchi and S.-K. Yang [EY]. The original formulation was inaccurate. The correct one was given by E. Getzler [Ge1] and independently by Y. Zhang [Z].

Apparently, the first proof was obtained by E. Getzler [Ge1] who applied two results – the unextended part of the Toda conjecture and the Virasoro constraints for $\mathbb{C}P^1$. The paper [DZ2] by B. Dubrovin – Y. Zhang contains a different proof based on the theory [DZ1] of integrable hierarchies associated to Frobenius structures and also uses the Virasoro constraints for $\mathbb{C}P^1$. The Virasoro constraints for $\mathbb{C}P^1$ were proved by A. Givental in [G3] by combining the fixed point localization formula [G4] for $\mathcal{D}_X$ with mirror symmetry and a certain loop group formalism. Our proof to the Toda Conjecture stays entirely within the paradigm developed in [G3] and pursued further in [G1, GM]. It relies directly on this formalism and the mirror model of $\mathbb{C}P^1$ as well as Kontsevich's theorem. Yet another approach to the Toda conjecture, due to A. Okounkov – R. Pandharipande [OP2](and based on fixed point localization but buy-passing Kontsevich's theorem), yields the *equivariant* version of the Toda conjecture [Ge2] as well as the unextended part of the non-equivariant Toda conjecture.

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2. From Ancestors to KdV

In this section we give a proof of Theorem 1.1.

2.1. **The HQE of $\mathbb{C}P^1$.**

The vertex oerators $\Gamma_\tau^{\pm\alpha}(\lambda)$ and Γ_τ^δ, and the function $c_\alpha(\lambda,\tau)$ depend on a choice of a path C connecting λ with the reference point λ_0. We want to show that the expression (1.7), when computed at $\widehat{w}'_\tau - \widehat{w}''_\tau \in \mathbb{Z}$, is independent of the choice of C.

Recall that the thimbles $\beta_i,\ i = 1, 2$ are determined by the paths $C_i,\ i = 1, 2$. We fix generators γ_i of the fundamental group $\pi_1(\mathbb{C}\backslash\{u_1, u_2\}, \lambda_0)$ as follows: γ_i is a path starting at λ_0 and approaching u_i along the path C_i; when sufficiently close

to u_i, γ_i makes a small circle (counter-clockwise) around u_i and it returns back to λ_0 along the path C_i. Denote $C' = C\circ\gamma_i$ and by $\beta_1'(\lambda)$ and $\beta_2'(\lambda)$ the Lefschetz thimbles corresponding to the new path C'. For simplicity, we will consider the case $i=2$ (the case $i=1$ is similar and simpler).

According to the Pickard–Lefschetz theorem, the change of the thimbles is measured by the following formula ([AGV], chapter 1, section 1.3):

$$\begin{aligned}\beta_1'(\lambda) &= \beta_1(\lambda) - \langle\beta_1,\beta_2\rangle\beta_2(\lambda) = \beta_1(\lambda)+2\beta_2(\lambda),\\ \beta_2'(\lambda) &= \beta_2(\lambda) - \langle\beta_2,\beta_2\rangle\beta_2(\lambda) = -\beta_2(\lambda).\end{aligned}$$

Thus

$$\begin{aligned}\alpha'(\lambda) &= \beta_1'/2 = -\alpha(\lambda)+\phi(\lambda),\\ \phi'(\lambda) &= \beta_1'(\lambda)+\beta_2'(\lambda) = \beta_1(\lambda)+\beta_2(\lambda) = \phi(\lambda)\end{aligned}$$

and we get $\mathbf{f}_\tau^{\pm\alpha'}(\lambda) = \mathbf{f}_\tau^{\mp\alpha}(\lambda)+\mathbf{f}_\tau^{\phi}(\lambda)$ and $\mathbf{f}_\tau^{\phi'}(\lambda)=\mathbf{f}_\tau^{\phi}(\lambda)$. In particular, we find that the vertex operator Γ_τ^δ does not depend on the choice of C. Let us compare $\Gamma_\tau^{\pm\alpha'}$ and $\Gamma_\tau^{\pm\alpha}$. Note that

$$(I_\phi^{(-1)}(\lambda,\tau),1) = \partial_\lambda\int_\phi(\lambda-f_t(x))\,dx/x = 2\pi i, \tag{2.1}$$

$$(I_\phi^{(-1)}(\lambda,\tau),P) = -\partial_t\int_\phi(\lambda-f_t(x))\,dx/x = 0. \tag{2.2}$$

Thus $I_\phi^{(k)}=0$ for all $k\geq 0$ i.e., $(\mathbf{f}_\tau^\phi)_+=0$ and we get $\Gamma_\tau^{\pm\alpha'}=\Gamma_\tau^{\pm\phi}\Gamma_\tau^{\mp\alpha}$.

Let us analyze the coefficients c_α. Introduce the following vectors in H^2:

$$\mathbf{1}_i = \frac{1}{\sqrt{\Delta_i}}\left(\phi^0+\frac{\partial u_i}{\partial t}\phi^1\right),\ i=1,2,$$

where, recall that $\{\phi_0=\mathbf{1},\phi_1=P\}$ is a basis of H, $\{\phi^0,\phi^1\}$ is its dual basis with respect to the Poincaré pairing, and $u_i(t)$ are the critical values of the function $f_t(x)=x+Qe^t/x$. The factors Δ_i are the Hessians of f_t at the critical point x_i with respect to the volume form ω.

Lemma 2.1. *For ξ sufficiently close to u_i the period vector $I_{\beta_i}^{(0)}$ can be expanded into a series of the following type*

$$I_{\beta_i}^{(0)}(\xi,\tau) = \frac{2}{\sqrt{2(\xi-u_i)}}\left(\mathbf{1}_i + A_{i,1}[2(\xi-u_i)]+A_{i,2}[2(\xi-u_i)]^2+\dots\right), \tag{2.3}$$

where the path C specifying $\beta_i(\xi)$ is the same as C_i everywhere in $\mathbb{C}\backslash\{u_1,u_2\}$, except for a small disk around u_i, where the two paths split – C_i leads to u_i and C leads to ξ.

[2] $\{\mathbf{1}_1,\mathbf{1}_2\}$ is a basis of H such that $(\mathbf{1}_i,\mathbf{1}_j)=\delta_{ij}$ and $\mathbf{1}_i\bullet_\tau\mathbf{1}_j=\delta_{ij}\mathbf{1}_i$, where $\bullet_\tau$ is the quantum cup product at the point $\tau=tP\in H$.

Proof. We follow [AGV], chapter 3, section 12, Lemma 2.

Let y be *a unimodular* coordinate on $\mathbb{C}^*$ for the volume form ω i.e., $\omega = dy$. The Taylor's expansion of f_t is:

$$f_t(y) = u_i + \frac{\Delta_i}{2}(y-y_i)^2 + \ldots,$$

where y_i is the y-coordinate of the critical point x_i. From this expansion we find that the equation $f_t(y) = \xi$ has two solutions in a neighborhood of $\xi = u_i$:

$$y_\pm = y_i \pm \frac{1}{\sqrt{\Delta_i}}\sqrt{2(\xi-u_i)} + \text{ h.o.t.},$$

where h.o.t. means higher order terms. Thus the integral in the definition of $I^{(0)}_{\beta_i}(\xi,\tau)$ has the following expansion:

$$\int_{\beta_i(\xi)} \omega = y_+(\xi) - y_-(\xi) = \frac{2}{\sqrt{\Delta_i}}\sqrt{2(\xi-u_i)} + \text{ h.o.t. }.$$

The last expansion yields:

$$(I^{(0)}_{\beta_i}(\xi,\tau),1) = \frac{2}{\sqrt{2(\xi-u_i)}}\,\frac{1}{\sqrt{\Delta_i}} + \text{ h.o.t. },$$

$$(I^{(0)}_{\beta_i}(\xi,\tau),P) = \frac{2}{\sqrt{2(\xi-u_i)}}\,\frac{1}{\sqrt{\Delta_i}}\,\frac{\partial u_i}{\partial t} + \text{ h.o.t. }.$$

The lemma follows. $\square$

Using Lemma 2.1 we get

$$c_{\alpha'}/c_\alpha = \exp\left(-\int_{\gamma_2}(I^{(0)}_\alpha(\xi,\tau), I^{(0)}_\alpha(\xi,\tau))d\xi\right) = \exp\left(-\frac{1}{2}\int_{\gamma_2}\frac{d\xi}{\xi-u_i}\right) = -1. \tag{2.4}$$

For the 2-nd equality in (2.4), we used that $I^{(0)}_\alpha = I^{(0)}_{\phi/2} - I^{(0)}_{\beta_2/2} = -I^{(0)}_{\beta_2/2}$ and that the expansion (2.3) holds (because γ_2 is a small loop around u_2) and only its leading term contributes to the integral.

Lemma 2.2. *Let $r \in \mathbb{Q}$. Then the following identity between operators acting on $\pi \otimes_{\mathcal{A}} \pi$ holds:*

$$\left(\Gamma^{\delta\#}_\tau \otimes \Gamma^\delta_\tau\right)\left(\Gamma^{r\,\phi}_\tau \otimes \Gamma^{-r\,\phi}_\tau\right) = e^{2\pi i(\hat{w}_\tau\otimes 1 - 1\otimes \hat{w}_\tau)\,r}\left(\Gamma^{\delta\#}_\tau \otimes \Gamma^\delta_\tau\right).$$

Proof. By definition (see formulas (2.1))

$$\mathbf{f}^\phi_\tau(\lambda) = 2\pi i\, P(-z)^{-1} + \sum_{k\geq 1} I^{(-1-k)}_\phi(\lambda,\tau)(-z)^{-k-1}.$$

Comparing with the definition of w_τ we get $\mathbf{f}^{\phi/2\pi i}_\tau(\lambda) \in w_\tau + z^{-1}\mathcal{H}_-$. Thus

$$\Omega(v_\tau, \mathbf{f}^{\phi/2\pi i}_\tau(\lambda)) = \Omega(v_\tau, w_\tau) = -1, \quad \Omega(w_\tau, \mathbf{f}^{\phi/2\pi i}_\tau(\lambda)) = 0\ .$$

For all $f, g \in \mathcal{H}$ we have:

$$e^{\hat{f}}e^{\hat{g}} = e^{[\hat{f},\hat{g}]}e^{\hat{g}}e^{\hat{f}} = e^{\Omega(\hat{f},\hat{g})}e^{\hat{g}}e^{\hat{f}},$$

because for linear Hamiltonians the quantization is a representation of Lie algebras. Thus

$$\Gamma^\delta_\tau \Gamma^{r\,\phi}_\tau = \exp\left((\widehat{\mathbf{f}}^{\phi/2\pi i}_\tau - \widehat{w}_\tau)(\epsilon\partial_x - \log\sqrt{Q})\right)\exp\left(\frac{x}{\epsilon}\widehat{v}_\tau\right)\exp\left(r\widehat{\mathbf{f}}^\phi_\tau\right) =$$
$$\exp\left(r\widehat{\mathbf{f}}^\phi_\tau\right)\exp\left((\widehat{\mathbf{f}}^{\phi/2\pi i}_\tau - \widehat{w}_\tau)(\epsilon\partial_x - \log\sqrt{Q})\right)\exp\left((rx/\epsilon)\Omega(v_\tau,\mathbf{f}^\phi_\tau)\right)\times$$
$$\exp\left(\frac{x}{\epsilon}\widehat{v}_\tau\right) = e^{2\pi i(\hat{w}_\tau-(x/\epsilon))r}\Gamma^\delta_\tau$$

Similarly, $\Gamma^{\delta\#}_\tau\Gamma^{r\,\phi}_\tau = e^{2\pi i\hat{w}_\tau\,r}\Gamma^{\delta\#}_\tau e^{-2\pi i x r/\epsilon}$. The Lemma follows. □

Since $\Gamma^{\pm\alpha'}_\tau = \Gamma^{\pm\phi}_\tau\Gamma^{\mp\alpha}_\tau$ and $c_{\alpha'} = -c_\alpha$, using Lemma 2.2, we get that the HQE of $\mathbb{C}P^1$ do not depend on the choice of the path C.

2.2. **Tame asymptotical functions.** The total ancestor potential has some special property which makes the expression (1.7) a formal series with coefficients meromorphic functions in λ.

An *asymptotical function* is, by definition, an expression

$$\mathcal{T} = \exp\sum_{g=0}^{\infty}\epsilon^{2g-2}\mathcal{T}^{(g)}(\mathbf{t};Q),$$

where $\mathcal{T}^{(g)}$ are formal series in the sequence of vector variables $t_0, t_1, t_2, \ldots$ with coefficients in the Novikov ring $\mathbb{C}[[Q]]$. Furthermore, $\mathcal{T}$ is called *tame* if

$$\frac{\partial}{\partial t_{k_1,a_1}}\cdots\frac{\partial}{\partial t_{k_r,a_r}}\bigg|_{\mathbf{t}=0}\mathcal{T}^{(g)} = 0 \quad \text{whenever} \quad k_1+k_2+\ldots+k_r > 3g-3+r,$$

where $t_{k,a}$ are the coordinates of t_k with respect to $\{\phi_0,\phi_1\}$. The total ancestor potential $\mathcal{A}_\tau$ is a tame asymptotical function, because the tameness conditions is trivially satisfied for dimensional reasons: $\dim\overline{\mathcal{M}}_{g,r} = 3g-3+r$.

Let $\mathcal{T}$ be a tame asymptotical function. The dilaton shift $\mathbf{t}(z) = \mathbf{q}(z)+z$ identifies $\mathcal{T}$ with an element of the Fock space. Let $\Gamma^\pm_i = \exp\left(\pm\sum I^{(n)}_i(\lambda)(-z)^n\right)$ be a finite set of vertex operators, where $I^{(n)}_i$ are meromorphic functions. Consider the expression

$$\sum_i c_i(\lambda)\left(\Gamma^+_i\otimes\Gamma^-_i\right)\left(\mathcal{T}(\mathbf{q}')\mathcal{T}(\mathbf{q}'')\right)d\lambda, \tag{2.5}$$

where c_i are meromorphic functions. According to [G1], Proposition 6, the tameness of $\mathcal{T}$ implies that (2.5), after the substitution $\mathbf{q}' = \mathbf{x}+\epsilon\mathbf{y}$, $\mathbf{q}'' = \mathbf{x}-\epsilon\mathbf{y}$ and after dividing by $\exp\left(2\mathcal{T}^{(0)}(\mathbf{x})/\epsilon^2\right)$, expands into a power series in ϵ, $\mathbf{x}$, and $\mathbf{y}$ whose coefficients depend polynomially on finitely many $I^{(n)}_i$.

In particular, (1.7) can be interpreted as a formal series in ϵ, $\mathbf{x}$, and $\mathbf{y}$ ($y_{0,0}$ excluded) with coefficients meromorphic functions in λ. The vertex operators could have poles only at the critical values u_1, u_2. Thus the regularity property follows if we prove that there are no poles at $\lambda = u_i$.

2.3. **The twisted loop group formalism.** Let

$$\mathcal{L}^{(2)}\mathrm{GL}(H) = \{M(z) \in \mathrm{GL}(\mathcal{H}) \mid M^*(-z)M(z) = 1\},$$

where $*$ means the transposition with respect to the Poincaré pairing, be the twisted loop group. The elements of the twisted loop group of the type $M = 1 + M_1 z + M_2 z^2 + \dots$ (respectively $M = 1 + M_1 z^{-1} + M_2 z^{-2} + \dots$) are called upper-triangular (respectively lower-triangular) linear transformations. They can be quantized as follows: write $M = \log A$, then $A(z)$ is an infinitesimal symplectic transformation. We define $\widehat{M} = \exp \hat{A}$, where A is identified with the quadratic Hamiltonian $\Omega(A\mathbf{f}, \mathbf{f})/2$ and on the space of quadratic Hamiltonians the quantization rule $\widehat{\ }$ is defined by:

$$(q_{k,\alpha}q_{l,\beta})\widehat{\ }:= \frac{q_{k,\alpha}q_{l,\beta}}{\epsilon^2}, \quad (q_{k,\alpha}p_{l,\beta})\widehat{\ }:= q_{k,\alpha}\frac{\partial}{\partial q_{l,\beta}}, \quad (p_{k,\alpha}p_{l,\beta})\widehat{\ }:= \epsilon^2 \frac{\partial^2}{\partial q_{k,\alpha}\partial q_{l,\beta}}.$$

We remark that $\widehat{\ }$ defines only *a projective representation* of the subgroups of lower-triangular and upper-triangular elements of $\mathcal{L}^{(2)}\mathrm{GL}(H)$ on the Fock space $\mathcal{B}$.

For the proof of Theorem 1.1 we will need to conjugate vertex operators with an upper-triangular linear transformation $R = 1 + R_1 z + R_2 z^2 + \dots \in \mathcal{L}^{(2)}\mathrm{GL}(H)$. The following formula holds ([G1], section 7):

$$\widehat{R}^{-1} e^{\hat{f}} \widehat{R} = e^{V f_-^2/2} \left(e^{R^{-1}f}\right)\widehat{\ }, \tag{2.6}$$

where $-$ means truncating the non-negative powers of z, $f_- = \sum_{k\geq 0}(-1)^{-1-k}(f_{-1-k}, q_k)$ is interpreted (via the symplectic form) as a linear function in $\mathbf{q}$, and $V(\partial, \partial) = \sum (V_{kl}\phi^a, \phi^b)\partial_{q_{l,a}}\partial_{q_{k,b}}$ is a second order differential operator whose coefficients are defined by

$$\sum_{k,l\geq 0} V_{kl} w^k z^l = \frac{1 - R(w)R^*(z)}{w+z}, \tag{2.7}$$

2.4. **The ancestor potential and mirror symmetry.** Following [G3] we will use the twisted loop group formalism to express the total ancestor potential in terms of oscillating integrals defined on the mirror model of $\mathbb{C}P^1$.

Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$. Equip the loop space $\mathbb{C}^2((z^{-1}))$ with a symplectic structure via (1.1), where the inner product in $\mathbb{C}^2$ is the standard one: $(e_i, e_j) = \delta_{ij}$. Denote $\mathcal{B}_{\mathbb{C}^2}$ the Bosonic Fock space which consists of functions defined in the formal neighborhood of $-e_1 - e_2$. In fact, $\mathbb{C}^2((z^{-1})) \cong \mathcal{H}_{\mathrm{pt}} \oplus \mathcal{H}_{\mathrm{pt}}$ and $\mathcal{B}_{\mathbb{C}^2} \cong \mathcal{B}_{\mathrm{pt}} \otimes \mathcal{B}_{\mathrm{pt}}$, where $\mathcal{H}_{\mathrm{pt}}$ and $\mathcal{B}_{\mathrm{pt}}$ are respectively the symplectic vector space and the Bosonic Fock space associated with $X = \mathrm{pt}$.

On the other hand, the map $\Psi(t) : \mathbb{C}^2 \to H$, defined by $\Psi(e_i) := \mathbf{1}_i$, is a linear isomorphism which respects the inner products in $\mathbb{C}^2$ and H. Thus $\Psi(t)$ induces isomorphisms $\Psi(t) : \mathbb{C}^2((z^{-1})) \to \mathcal{H}$ and $\widehat{\Psi} : \mathcal{B}_{\mathrm{pt}} \otimes \mathcal{B}_{\mathrm{pt}} \to \mathcal{B}$, where $\widehat{\Psi}(G_1 \otimes G_2)(\mathbf{q}) := G_1(\mathbf{q}^1)G_2(\mathbf{q}^2)$, $\mathbf{q}^1$ and $\mathbf{q}^2$ are defined by $\Psi^{-1}\mathbf{q} = \mathbf{q}^1 e_1 + \mathbf{q}^2 e_2$. According to [G3], the ancestor potential is given by the following formula:

$$\mathcal{A}_\tau = \widehat{\Psi} \widehat{R} e^{\widehat{U}/z} (\mathcal{D}_{\mathrm{pt}} \otimes \mathcal{D}_{\mathrm{pt}}), \tag{2.8}$$

where R and $e^{U/z}$ are certain respectively upper-triangular and lower-triangular linear transformations in $\mathcal{L}^{(2)}\mathrm{GL}(\mathbb{C}^2)$.

The factor $e^{U/z}$ in formula (2.8) is redundant because $\mathcal{D}_{\mathrm{pt}}$ satisfies the string equation ([G3]). It will be explained in subsection 3.1 that $\mathcal{D}_{\mathrm{pt}} = \mathcal{A}_{\mathrm{pt}}$. Also, $\widehat{\Psi}$ intertwines the action of the twisted loop groups $\mathcal{L}^{(2)}\mathrm{GL}(\mathbb{C}^2)$ and $\mathcal{L}^{(2)}\mathrm{GL}(H)$ i.e., $\widehat{\Psi}\widehat{R}\widehat{\Psi}^{-1} = (\Psi R \Psi^{-1})\widehat{}$. Thus formula (2.8) is equivalent to:

$$\mathcal{A}_\tau(\mathbf{q}) = \widehat{R}\left(\mathcal{A}_{\mathrm{pt}}(\mathbf{q}^1)\mathcal{A}_{\mathrm{pt}}(\mathbf{q}^2)\right), \tag{2.9}$$

where $R \in \mathcal{L}^{(2)}\mathrm{GL}(H)$ is a certain upper-triangular linear transformation and $\mathbf{q}^i$ are the coordinates of $\mathbf{q}$ with respect to the basis $\{\mathbf{1}_1, \mathbf{1}_2\}$.

The linear transformation R can be expressed in terms of oscillating integrals. Let $\beta_i(\infty) \subset \mathbb{C}^*$ be an extension of the Lefschetz thimble β_i along a path $C(\infty)$ starting at λ_0 and approaching $\lambda = \infty$ in such a way that $\mathrm{Re}\,\lambda < 0$. Then the oscillating integrals

$$\mathcal{J}_i(t,z) = (-2\pi z)^{-1/2} \int_{\beta_i(\infty)} e^{f_t(x)/z}\omega, \quad i = 1, 2, \tag{2.10}$$

are well defined and according to [G3] the matrix

$$J = \begin{bmatrix} \mathcal{J}_1 & \mathcal{J}_2 \\ z\partial_t \mathcal{J}_1 & z\partial_t\mathcal{J}_2 \end{bmatrix} \cdot \begin{bmatrix} e^{-u_1/z} & 0 \\ 0 & e^{-u_2/z}\end{bmatrix}$$

is asymptotic as $z \to 0$ to the matrix of the linear operator R with respect to the basises $\{\mathbf{1}_1, \mathbf{1}_2\}$ and $\{\phi^0, \phi^1\}$ respectively in the domain and the co-domain of R.

2.5. **Conjugating vertex operators.** Let λ be sufficiently close to the critical value u_i and assume that the path C specifying the vertex operator $\Gamma_\tau^{\beta_i/2}$ is the same as in Proposition 2.1. According to (2.6), in order to conjugate the vertex operator $\Gamma_\tau^{\beta_i/2}$ by the symplectic transformation R, we need to derive formulas for the vector $R^{-1}\mathbf{f}_\tau^{\beta_i/2}$ and for the phase factor $V(\mathbf{f}_\tau^{\beta_i/2})^2_-/2$. The computations are essentially the same as in [G1].

Lemma 2.3. *a) The period vectors satisfy:* $\partial_\lambda I^{(n)}_{\beta_i} = I^{(n+1)}_{\beta_i}$,

b) For every $k \geq 0$, the following formula holds:

$$(-z)^{k+3/2}\mathcal{J}_i = \frac{1}{\sqrt{2\pi}} \int_{u_i}^{\infty} e^{\lambda/z}\left(\int_{\beta_i(\lambda)} \frac{(\lambda - f_t)^k}{k!}\omega\right) d\lambda,$$

where the integration path is $C(\infty)\circ C_i^{-1}$.

Proof. a) Note that

$$\int_{\beta_i(\lambda)} \frac{(\lambda - f_\tau(x))^{k+1}}{(k+1)!}\omega = \int_{\lambda_0}^{\lambda}\left(\int_{\partial\beta_i(\xi)} \frac{(\lambda - f_\tau(x))^{k+1}}{(k+1)!}\frac{\omega}{df_t}\right) d\xi.$$

Differentiating with respect to λ we get

$$\partial_\lambda \int_{\beta_i(\lambda)} \frac{(\lambda - f_\tau(x))^{k+1}}{(k+1)!}\omega = \int_{\lambda_0}^{\lambda}\left(\int_{\partial\beta_i(\xi)} \frac{(\lambda - f_\tau(x))^{k}}{k!}\frac{\omega}{df_t}\right) d\xi + $$
$$+ \int_{\partial\beta_i(\lambda)} \frac{(\lambda - f_\tau(x))^{k+1}}{(k+1)!}\frac{\omega}{df_t} = \int_{\beta_i(\lambda)} \frac{(\lambda - f_\tau(x))^{k}}{k!}\omega,$$

where we used that the function $\lambda - f_t(x)$ vanishes on the boundary of the cycle $\beta_i(\lambda) \in H_1(\mathbb{C}^*, f_t^{-1}(\lambda); \mathbb{Z})$. Part a) follows.

b) By definition,

$$\begin{aligned}(-2\pi z)^{1/2}\mathcal{J}_i &= \int_{u_i}^{-\infty} e^{\lambda/z}\int_{\partial\beta_i(\lambda)} \frac{\omega}{df_t} d\lambda = \int_{u_i}^{-\infty} e^{\lambda/z}\partial_\lambda\left(\int_{\partial\beta_i(\lambda)} d^{-1}\omega\right) d\lambda = \\ & \int_{u_i}^{-\infty} e^{\lambda/z}\partial_\lambda\left(\int_{\beta_i(\lambda)} \omega\right) d\lambda = -z^{-1}\int_{u_i}^{-\infty} e^{\lambda/z}\left(\int_{\beta_i(\lambda)} \omega\right) d\lambda.\end{aligned}$$

Part b) follows from a) and integration by parts. □

Lemma 2.4. *The following formula holds:*

$$R^{-1}\mathbf{f}_\tau^{\beta_i/2} = \sum_{n\in\mathbb{Z}}(-z\partial_\lambda)^n \frac{\mathbf{1}_i}{\sqrt{2(\lambda-u_i)}}, \tag{2.11}$$

where (for $n<0$) ∂_λ^{-1} means integration and the corresponding integration constants are "set to 0".

Proof. Let $J_i = e^{-u_i/z}\mathcal{J}_i\phi^0 + e^{-u_i/z}(z\partial_t\mathcal{J}_i)\phi^1$. Using Lemma 2.3 b) we get

$$J_i = (-2\pi z)^{-1/2}\int_{u_i}^{-\infty} e^{(\lambda-u_i)/z} I^{(0)}_{\beta_i}(\lambda,\tau) d\lambda.$$

From here our argument is the same as the proof of Theorem 3 in [G1]. Near the critical value the period $I^{(0)}_{\beta_i}(\lambda,\tau)$ has the expansion (2.3). Using the change of variables $\lambda - u_i = -zx^2/2$ we compute

$$\begin{aligned}\frac{2}{\sqrt{-2\pi z}}\int_{u_i}^{-\infty} e^{(\lambda-u_i)/z}[2(\lambda-u_i)]^{k-1/2}d\lambda &= (-z)^k\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}x^{2k}dx = \\ &= (-z)^k(2k-1)!!\ .\end{aligned}$$

Thus J_i has the following asymptotic

$$J_i \sim \sum_{k=0}^{\infty}(2k-1)!!\, A_{i,k}(-z)^k.$$

Since, by definition, the asymptotic of J_i is $R\mathbf{1}_i$ we get $A_{i,k} = (-1)^k R_k/(2k-1)!!$. Thus

$$\mathbf{f}_\tau^{\beta_i}(\lambda) = \sum_{n\in\mathbb{Z}}(-z\partial_\lambda)^n I^{(0)}_{\beta_i}(\lambda,\tau) = 2\sum_{n\in\mathbb{Z}}\sum_{k=0}^{\infty}(-z\partial_\lambda)^n(-1)^k R_k\mathbf{1}_i\frac{[2(\lambda-u_i)]^{k-1/2}}{(2k-1)!!} =$$

$$= \sum_{n\in\mathbb{Z}} (-z\partial_\lambda)^n R_k \mathbf{1}_i (-\partial_\lambda)^{-k} \frac{1}{\sqrt{2(\lambda-u_i)}} = 2R \sum_{n\in\mathbb{Z}} (-z\partial_\lambda)^n \frac{\mathbf{1}_i}{\sqrt{2(\lambda-u_i)}}.$$

□

Lemma 2.5. *Let V be the quadratic form* (2.7)*. Then*

$$V\mathbf{f}_-^2 = -\lim_{\epsilon\to 0} \int_\lambda^{u_i+\epsilon} \left(\Big(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\Big) - \frac{1}{2(\xi-u_i)}\right) d\xi, \tag{2.12}$$

where $\mathbf{f} := \mathbf{f}_\tau^{\beta_i/2}$ and the integration path is $C_i(\epsilon)\circ C^{-1}$.

Proof. The proof is taken from [G1], page 490. When $\mathbf{f} = \sum_{k\in\mathbb{Z}} I_{\beta_i/2}^{(k)}(-z)^k$ we have $\mathbf{f}_- = \sum_{k\geq 0} (I_{\beta_i/2}^{(-1-k)}, \phi^a) q_{k,a}$. Using $\partial_\lambda I_{\beta_i}^{(-1-k)} = I_{\beta_i}^{(-k)}$, we find

$$\begin{aligned} \partial_\lambda V\mathbf{f}_-^2 &= \frac{1}{4}\sum_{k,l\geq 0} \partial_\lambda \left(V_{k,l} I_{\beta_i}^{(-1-l)}, I_{\beta_i}^{(-1-k)}\right) = \frac{1}{4}\sum_{k,l\geq 0} \left([V_{k-1,l}+V_{k,l-1}] I_{\beta_i}^{(-l)}, I_{\beta_i}^{(-k)}\right) = \\ &= \frac{1}{4}(I_{\beta_i}^{(0)}, I_{\beta_i}^{(0)}) - \frac{1}{4}(\sum_{l\geq 0} R_l^* I_{\beta_i}^{(-l)}, \sum_{k\geq 0} R_l^* I_{\beta_i}^{(-k)}). \end{aligned}$$

On the other hand, $R^*(z) = R^{-1}(-z)$ because $R\in\mathcal{L}^{(2)}\mathrm{GL}(H)$. Thanks to Lemma 2.4 $\sum_{k\geq 0} R_l^* I_{\beta_i}^{(-k)} = 2\mathbf{1}_i/\sqrt{2(\lambda-u_i)}$. Also $V\mathbf{f}_-^2 = 0$ at $\lambda = u_i$ because $I_{\beta_i}^{(-1-k)} = 2[2(\lambda-u_i)]^{k+1/2}(\mathbf{1}_i + \ldots)$ vanish at $\lambda = u_i$. The lemma follows. □

2.6. **Proof of Theorem 1.1.** According to the discussion in subsection 2.2, it is enough to show that the HQE (1.7) has no pole at $\lambda = u_i,\ i = 1,2$. Let us assume that λ is close to the critical value u_i and that the path C between λ_0 and λ is the same as in Lemma 2.1. There are two cases:

Case 1. When $i = 1$ i.e., λ is close to u_1. Using formula (2.9) for the ancestor potential, the conjugation formula (2.6), Lemma 2.4, and Lemma 2.5, we transform the HQE (1.7) into

$$\begin{aligned} &\left(\Gamma_\tau^{\delta\#}\otimes\Gamma_\tau^{\delta}\right)\left(\widehat{R}\otimes\widehat{R}\right) \\ &\quad \left[b_\alpha \left(\Gamma_{u_1}^+\otimes\Gamma_{u_1}^- - \Gamma_{u_1}^-\otimes\Gamma_{u_1}^+\right) \left(\mathcal{A}_{\mathrm{pt}}(\mathbf{q}^1)\otimes\mathcal{A}_{\mathrm{pt}}(\mathbf{q}^1)\right)\ d\lambda\right] \left(\mathcal{A}_{\mathrm{pt}}(\mathbf{q}^2)\otimes\mathcal{A}_{\mathrm{pt}}(\mathbf{q}^2)\right), \end{aligned} \tag{2.13}$$

where $\Gamma_{u_1}^{\pm}$ are the vertex operators (1.3) and the function b_α is given by

$$b_\alpha = c_\alpha e^{V\mathbf{f}_-^2} = \lim_{\epsilon\to 0}\exp\left(-\int_1^\epsilon \frac{d\xi}{2\xi} + \int_\lambda^{u_1+\epsilon}\frac{d\xi}{2(\xi-u_1)}\right) = \exp\left(\int_\lambda^{u_1+1}\frac{d\xi}{2(\xi-u_1)}\right).$$

Note that the expression in the [ ]-brackets in (2.13) is precisely the HQE (1.4) of the ancestor potential of a point. Thus it is regular in λ. On the other hand $I_\phi^{(n)}(\lambda,\tau)$ is polynomial in λ if $n<0$ and it is 0 otherwise (see (2.1) and (2.2)). Thus Γ_τ^δ is regular in λ.

Case 2. When $i = 2$ i.e., λ is close to u_2. We reduce this case to the previous one. Let $\alpha' = \beta_2/2$ and

$$c_{\alpha'} = \lim_{\epsilon\to 0}\exp\left(\int_\lambda^{u_2+\epsilon}\Big(I_{\alpha'}^{(0)}(\xi,\tau), I_{\alpha'}^{(0)}(\xi,\tau)\Big)d\xi - \langle\alpha',\alpha'\rangle\int_1^\epsilon\frac{d\xi}{\xi}\right).$$

We will prove that $c_\alpha(\lambda)/c_{\alpha'}(\lambda)$ is a constant depending only on the reference point λ_0 and the paths C_1 and C_2.

For every $\chi = r_1\beta_1 + r_2\beta_2,\ r_1, r_2 \in \mathbb{Q}$ let $\mathcal{W}_\chi = (I^{(0)}_\chi(\xi,\tau), I^{(0)}_\chi(\xi,\tau))d\xi$ – it is a meromorphic 1-form with simple poles at $\xi = u_1$ and $\xi = u_2$ (see Lemma 2.1). Let $C_i(\epsilon)$ be the path starting at λ_0 and reaching the point $u_i+\epsilon$ along C_i. Then (2.14)

$$c_\alpha(\lambda)/c_{\alpha'}(\lambda) = \lim_{\epsilon\to 0}\exp\left(\int_{C_1(\epsilon)\circ C}\mathcal{W}_\alpha - \int_{C_2(\epsilon)\circ C}\mathcal{W}_{\alpha'}\right) = \lim_{\epsilon\to 0}\exp\left(\int_{C_1(\epsilon)\circ C_2^{-1}(\epsilon)}\mathcal{W}_\alpha\right),$$

where for the second equality we used that $\mathcal{W}_\alpha = \mathcal{W}_{\alpha'}$ which follows from $\alpha = \phi/2 - \alpha'$ and $I^{(0)}_\phi = 0$.

On the other hand $\Gamma^{\pm\alpha}_\tau = \Gamma^{\pm\phi/2}_\tau\Gamma^{-\alpha'}_\tau$. According to Lemma 2.2,

$$\left(\Gamma^{\delta\#}_\tau\otimes\Gamma^{\delta}_\tau\right)\left(\Gamma^{\pm\phi/2}_\tau\otimes\Gamma^{\mp\phi/2}_\tau\right) = e^{\pm\pi i(\hat{w}'_\tau-\hat{w}''_\tau)}\Gamma^{\delta\#}_\tau\otimes\Gamma^{\delta}_\tau = e^{\pi i(\hat{w}'_\tau-\hat{w}''_\tau)}\Gamma^{\delta\#}_\tau\otimes\Gamma^{\delta}_\tau,$$

where the second equality holds whenever $\widehat{w}'_\tau - \widehat{w}''_\tau \in \mathbb{Z}$. Thus the HQE (1.7) are equivalent to the similar HQE, with α' instead of α.

The proof that the HQE corresponding to α' are regular at $\lambda = u_2$ is the same as in the previous case. □

3. From Ancestors to Descendants

In this section we give a proof of Corollary 1.2.

3.1. **Descendants and ancestors.** Let $S_\tau(z) = 1 + S_1z^{-1} + S_2z^{-2} + \ldots$ be the operator series defined by

$$(S_\tau\phi_\alpha,\phi_\beta) = (\phi_\alpha,\phi_\beta) + \sum_{k\geq 0}\langle\phi_\alpha\psi^k,\phi_\beta\rangle_{0,2}(\tau)z^{-1-k}. \tag{3.1}$$

It is a basic fact in quantum cohomology theory that S_τ is a lower-triangular linear transformation from $\mathcal{L}^{(2)}\mathrm{GL}(H)$ (see [G3], section 6 and the references there in). According to [CG], Appendix 2,

$$\mathcal{D}_X = e^{\mathrm{F}^{(1)}(\tau)}\widehat{S}_\tau^{-1}\mathcal{A}_\tau, \tag{3.2}$$

where $\mathrm{F}^{(1)} = \mathcal{F}^{(1)}\big|_{t_0=\tau, t_1=t_2=\ldots=0}$ is the genus-1 no-descendants potential. The action of S on a function G from the Fock space $\mathcal{B}$ is given by the following formula ([G3], Proposition 5.3):

$$\widehat{S}_\tau^{-1}G(\mathbf{q}) = e^{W(\mathbf{q},\mathbf{q})/2\epsilon^2}G([S_\tau\mathbf{q}]_+), \tag{3.3}$$

where the quadratic form $W(\mathbf{q},\mathbf{q}) = \sum(W_{kl}q_l,q_k)$ is defined by

$$W_{kl}w^{-k}z^{-l} = \frac{S^*_\tau(w)S_\tau(z)-1}{w^{-1}+z^{-1}}. \tag{3.4}$$

When $X = \mathrm{pt}$, $S_\tau = e^{\tau/z}$ and the genus-1 no-descendants potential $\mathrm{F}^{(1)}$ vanishes for dimensional reasons. Also, according to the string equation $(1/z)\widehat{\ }\mathcal{D}_{\mathrm{pt}} = 0$. Thus formula (3.2) yields $\mathcal{A}_{\mathrm{pt},\tau} = \mathcal{D}_{\mathrm{pt}}$.

The proof of Corollary 1.2 amounts to conjugating the vertex operators in the HQE (1.7) by S_τ. We will use the following formula ([G1], formula (17)):

$$\widehat{S}_\tau^{-1} e^{\hat{f}} \hat{S}_\tau = e^{-W((S_\tau^{-1}f)_+,(S_\tau^{-1}f)_+)/2} e^{(S_\tau^{-1}f)^{\hat{}}}, \tag{3.5}$$

where $+$ means truncating the terms corresponding to the negative powers of z. The key ingredient in the computation is that $\mathbf{f}_\tau^\alpha$ and S_τ satisfy the same system of differential equations with respect to the parameter τ.

3.2. **The small quantum differential equation.** Let us assume that the parameter $\tau = t\,P$. The quantum multiplication by P is, by definition, a linear operator on H, defined as follows:

$$(P \bullet_\tau \phi_i\,,\,\phi_j) := \langle P, \phi_i, \phi_j \rangle_{0,3}\,(\tau)$$

The linear operator $P\bullet_\tau$ is self-adjoint with respect to the Poincaré pairing — the correlator is symmetric, so we can switch ϕ_i and ϕ_j. According to [HKKPTVVZ], formula (26.15), $P\bullet_\tau$ has matrix $\begin{bmatrix} 0 & Qe^t \\ 1 & 0 \end{bmatrix}$ with respect to the basis $\{1, P\}$.

The ordinary differential equation

$$z\partial_t \Phi(t) = (P\bullet_\tau)\Phi(t), \quad \Phi(t) \in H \tag{3.6}$$

is called *the small quantum differential equation* of $\mathbb{C}P^1$. It admits a solution in terms of Gromov–Witten invariants of $\mathbb{C}P^1$ — the operator series S_τ is a fundamental solution to (3.6) ([HKKPTVVZ], Proposition 28.0.2).

Part of the mirror symmetry phenomena is that the small quantum differential equation can be solved in terms of the oscillating integrals (2.10) of the mirror model of $\mathbb{C}P^1$. Let $J_i(t,z)$, $i = 1,2$ be vector-valued functions with values in H defined by:

$$(J_i, \mathbf{1}) = \mathcal{J}_i, \quad (J_i, P) = z\partial_t \mathcal{J}_i,$$

where $\mathcal{J}_i$ are the oscillating integrals (2.10).

Lemma 3.1. *Vectors* J_i, $i = 1,2$ *are solutions to the small quantum differential equation* (3.6)*. They satisfy the following homogeneity condition:*

$$(z\partial_z + E)J_i(t,z) = \mu J_i(t,z),$$

where $E = 2\partial_t$ *is* the Euler vector field *and* μ *is* the Hodge grading operator *i.e., in the basis* $\{\mathbf{1}, P\}$, $\mu = \mathrm{diag}\{1/2, -1/2\}$.

Proof. For the first part of the lemma see [G3], section 10 (or just differentiate and use integration by parts).

Let us prove the homogeneity condition. Using that $J_i = (J_i, \mathbf{1})P + (J_i, P)\mathbf{1} = \mathcal{J}_i P + (z\partial_t \mathcal{J}_i)\mathbf{1}$ we see that the equation is equivalent to

$$(z\partial_z + E)\mathcal{J}_i = -(1/2)\mathcal{J}_i \quad \text{and} \quad (z\partial_z + E)z\partial_t \mathcal{J}_i = (1/2)z\partial_t \mathcal{J}_i.$$

It is enough to prove the 1-st equation because the second one can be derived from the 1-st one.

$$(z\partial_z + E)\mathcal{J}_i = (z\partial_z + 2\partial_t)(-2\pi z)^{-1/2}\int_{\beta_i(\infty)} \exp\left(f_t/z\right) dx/x =$$
$$(-2\pi z)^{-1/2}\int_{\beta_i(\infty)} \left(-(1/2) + z^{-1}(-f_t + 2Qe^t/x)\right)\exp\left(f_t/z\right) dx/x =$$
$$(-1/2)\mathcal{J}_i + (-2\pi z)^{-1/2}\int_{\beta_i(\infty)} d\exp\left(f_t/z\right) = (-1/2)\mathcal{J}_i.$$

□

Lemma 3.2. *The series $\mathbf{f}_\tau^{\beta_i}$ satisfies the small quantum differential equation* (3.6) *and the homogeneity condition:*

$$(z\partial_z + \lambda\partial_\lambda + E)\mathbf{f}_\tau^{\beta_i} = (\mu - 1/2)\mathbf{f}_\tau^{\beta_i} \tag{3.7}$$

Proof. We will show that $\mathbf{f}_\tau^{\beta_i}$ is a solution to (3.6). The derivation of (3.7) is similar.

The series $\mathbf{f}_\tau^{\beta_i}$ satisfies (3.6) if and only if $\partial_t I_{\beta_i}^{(n)} = -(P\bullet_\tau)I_{\beta_i}^{(n+1)}$. It is enough to prove that the last equality holds for every $n = -k-1$, $k \geq 0$ — according to Lemma 2.3, a), for $n = k$, $k \geq 0$ we need only to differentiate $k+1$ times, the equality corresponding to $n = -1$. Thus we need to prove that

$$(\partial_t I_{\beta_i}^{(-1-k)}, \mathbf{1}) = -(P\bullet_\tau I_{\beta_i}^{(-k)}, \mathbf{1}) \quad \text{and} \quad (\partial_t I_{\beta_i}^{(-1-k)}, P) = -(P\bullet_\tau I_{\beta_i}^{(-k)}, P) \tag{3.8}$$

On the other hand, by definition,

$$I_{\beta_i}^{(-1-k)} = (I_{\beta_i}^{(-1-k)}, \mathbf{1})P + (I_{\beta_i}^{(-1-k)}, P)\mathbf{1} = \left(\partial_\lambda \mathcal{I}_i^{(k+1)}\right)P + \left(-\partial_t\mathcal{I}_i^{(k+1)}\right)\mathbf{1},$$

where

$$\mathcal{I}_i^{(k+1)} = \int_{\beta_i(\lambda)} \frac{(\lambda - f_t)^{k+1}}{(k+1)!}\omega.$$

Thus after a short computation, we get that the differential equations (3.8) are equivalent to a single differential equation:

$$-\partial_t^2\mathcal{I}_i^{(k+1)} = (P\bullet_\tau P, \mathbf{1})\partial_t\partial_\lambda\mathcal{I}_i^{(k+1)} - (P\bullet_\tau P, P)\partial_\lambda^2\mathcal{I}_i^{(k+1)}. \tag{3.9}$$

Similarly, the fact that J_i is a solution to the small quantum differential equation is equivalent to a single differential equation for the oscillating integral $\mathcal{J}_i$:

$$\partial_t^2\mathcal{J}_i = (P\bullet_\tau P, \mathbf{1})(1/z)\partial_t\mathcal{J}_i + (P\bullet_\tau P, P)(1/z^2)\mathcal{J}_i.$$

According to Lemma 2.3 b), $(-z)^{k+5/2}\mathcal{J}_i$ is a Laplace transform along the path from $\mathcal{I}_i^{(k+1)}$. Thus the last equation implies (3.9) because under the Laplace transform multiplication by $1/z$ corresponds to $-\partial/\partial\lambda$. □

3.3. **The period vectors in a neighborhood of $\lambda = \infty$.** Let us assume that the reference point λ_0 belongs to a neighborhood of ∞, i.e. λ_0 is outside a disk containing the two critical points u_1 and u_2. Let λ and the path C specifying $\beta_i(\lambda)$, $i = 1, 2$ belong to this neighborhood of ∞ as well. We will show that each period vector expands into a series of the following type:

$$\Big(\sum_{j=0}^{N} A_j(\tau)\lambda^j \Big) \log \frac{\lambda}{\sqrt{Q}} + \sum_{k\geq 0} B_k(\tau)\lambda^{-k} \ , \tag{3.10}$$

where the coefficients are vectors in H which depend polynomially on t and Qe^t.

The equation $f_t(x) = \lambda$ has two solutions near $\lambda = \infty$, which can be expanded into Laurent series in λ^{-1} as follows:

$$\begin{aligned} x_+ &= \lambda + a_0 + a_1\lambda^{-1} + \dots \\ x_- &= Qe^t\lambda^{-1} + b_2\lambda^{-2} + b_3\lambda^{-3} + \dots, \end{aligned}$$

where the coefficients are polynomials in t and Qe^t.

The boundary of the cycle $\beta_i(\lambda) \in H_1(\mathbb{C}^*, f_t^{-1}(\lambda); \mathbb{Q})$ is given by:

$$\partial\beta_i(\lambda) = \pm\frac{1}{2}([x_+] - [x_-]), \tag{3.11}$$

where the sign depends on the choice of the path C_i (the sign does not depend on C!). Let us assume that the sign is $+$ for $i = 1$ and $-$ for $i = 2$, otherwise the argument is similar. Using the definition we find the following formula:

$$I^{(-1)}_{\beta_i}(\lambda, \tau) = \pm\frac{1}{2} \log \frac{x_+}{x_-} P \ \pm \frac{1}{2}(x_+ - x_-)\, \mathbf{1}. \tag{3.12}$$

Substituting the Laurent expansions of x_+ and x_- in (3.12) we see that, near $\lambda = \infty$, the period $I^{(-1)}_\alpha$ expands into a series of the type (3.10).

The period vector $I^{(n)}_{\beta_i}$, $n > 0$ has an expansion of type (3.10) obtained by differentiating $n+1$ times (with respect to λ) the expansion of $I^{(-1)}_{\beta_i}$. When $n < -1$, due to Lemma 3.2, the following recursive relation holds:

$$(\lambda - 2P\bullet)\, I^{(-k)}_{\beta_i}(\lambda, \tau) = (\mu + k + \frac{1}{2}) I^{(-k-1)}_{\beta_i}(\lambda, \tau). \tag{3.13}$$

When $k \geq 1$ the linear operator $\mu + k + (1/2)$ is invertible. Thus $I^{(-k-1)}_{\beta_i}$ can be expressed in terms of $I^{(-k)}_{\beta_i}$. Arguing by induction we find that $I^{(n)}_{\beta_i}$, $n < 0$ expands into a series of type (3.10).

Denote

$$\mathbf{f}^{\beta_i}(\lambda) = \sum_n I^{(n)}_{\beta_i}(\lambda)(-z)^n := S_\tau^{-1}\mathbf{f}^{\beta_i}_\tau(\lambda).$$

Each coefficient $I^{(n)}_{\beta_i}(\lambda)$ is a formal series of type (3.10) with coefficients independent of τ. The last statement follows from the fact that $\mathbf{f}^{\beta_i}_\tau$ and S_τ satisfy the same

differential equation with respect to t. Furthermore, using that $S_\tau^{-1}(z) = S_\tau^*(-z)$, we get

$$I_{\beta_i}^{(n)}(\lambda) = I_{\beta_i}^{(n)}(\lambda,\tau) + S_1^* I_{\beta_i}^{(n+1)}(\lambda,\tau) + \ldots = \left(1 + S_1^*\partial_\lambda + S_2^*\partial_\lambda^2 + \ldots\right) I_{\beta_i}^{(n)}(\lambda,\tau).$$

The last expression is a series of type (3.10) whenever $I_{\beta_i}^{(n)}(\lambda,\tau)$ is such.

For every $\alpha = r_1\beta_1 + r_2\beta_2$, $r_1, r_2 \in \mathbb{Q}$ define

$$I_\alpha^{(n)}(\lambda) = r_1 I_{\beta_1}^{(n)}(\lambda) + r_2 I_{\beta_2}^{(n)}(\lambda), \quad \mathbf{f}^\alpha(\lambda) = \sum_n I_\alpha^{(n)}(\lambda)(-z)^n, \quad \Gamma^\alpha(\lambda) = e^{\hat{\mathbf{f}}^\alpha(\lambda)}.$$

Lemma 3.3. *Let $\alpha = \beta_1/2$. Then the following formulas hold:*

$$I_\alpha^{(-1-n)}(\lambda) = \frac{\lambda^n}{n!}(\log\frac{\lambda}{\sqrt{Q}} - \mathcal{C}_n)P + \frac{\lambda^{n+1}}{2(n+1)!},\ n \geq 0\ ,$$
$$I_\alpha^{(0)}(\lambda) = \frac{1}{\lambda}P + \frac{1}{2}, \qquad I_\alpha^{(n)}(\lambda) = \frac{(-1)^n n!}{\lambda^{n+1}}P,\ n > 0\ .$$

Proof. The coefficients $S_k, k \geq 1$ in front of the powers of z in the series S_τ depend polynomially on t and Qe^t and after letting $t = Qe^t = 0$ they all vanish (see [HKKPTVVZ], Exercise 28.1.1.). Thus:

$$\mathbf{f}^\alpha(\lambda) = \mathbf{f}^\alpha(\lambda,\tau)|_{t=Qe^t=0}.$$

In particular we find:

$$I_\alpha^{(-1)}(\lambda) = \log\frac{\lambda}{\sqrt{Q}}\,P + \frac{\lambda}{2}\,\mathbf{1}. \tag{3.14}$$

The other vectors $I_\alpha^{(n)}$ can be easily obtained from $I_\alpha^{(-1)}$. Indeed, for $n = k \geq 0$ differentiate $k+1$ times (3.14) and for $n = -k-1, k \geq 1$ use the recursive relation (3.13) specialized to $t = Qe^t = 0$:

$$(\lambda - 2P\cup)I_\alpha^{(-k)} = (\mu + k + 1/2)I_\alpha^{(-k-1)},$$

where $P\cup$ is the classical cup product multiplication by P in the cohomology algebra H. $\square$

The phase factor $W_\tau(\mathbf{f}_+^\alpha(\lambda), \mathbf{f}_+^\alpha(\lambda))$ can be computed as follows:

$$\frac{d}{d\xi}W_\tau(\mathbf{f}_+^\alpha(\xi), \mathbf{f}_+^\alpha(\xi)) = -\sum_{k,l\geq 0}\Big((W_{k,l-1} + W_{k-1,l})(-1)^l I_\alpha^{(l)}(\xi), (-1)^k I_\alpha^{(k)}(\xi))\Big) =$$
$$-\sum_{k,l\geq 0}\Big(S_l(-1)^l I_\alpha^{(l)}(\xi), S_k(-1)^k I_\alpha^{(k)}(\xi))\Big) + \Big(I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi)\Big) =$$
$$-\Big(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\Big) + \Big(I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi)\Big)$$

where for the first equality we used Lemma 2.3 a) and for the second one – the definition (3.4) of the quadratic form W_τ. Since $\mathbf{f}_+^\alpha = (1/2)\mathbf{1}$ at $\xi = \infty$ we get

$$W_\tau(\mathbf{f}_+^\alpha(\lambda), \mathbf{f}_+^\alpha(\lambda)) = \frac{1}{4}W_\tau(\mathbf{1},\mathbf{1}) + \int_\lambda^\infty \Big((I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)) - (I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi))\Big)\,d\xi,$$

where the integration path is $C(\infty)\circ C^{-1}$.

The term $W_\tau(\mathbf{1},\mathbf{1})$ can be computed as follows:

$$W_\tau(\mathbf{1},\mathbf{1}) = (W_{0,0}\mathbf{1},\mathbf{1}) = (S_1\mathbf{1},\mathbf{1}) = \sum_{d,n}\frac{Q^d}{n!}\langle\mathbf{1},\mathbf{1},\tau,\dots,\tau\rangle_{0,n+2,d} =$$
$$\langle\mathbf{1},\mathbf{1},\tau\rangle_{0,3,0} = \int_{[\mathbb{C}P^1]}\tau = t,$$

where thanks to the string equation all terms in the sum on the first line, except the ones corresponding to $d=0$ and $n=1$, vanish.

Lemma 3.4. *The following formula holds:*

$$c_\alpha(\lambda)\left(\Gamma_\tau^{\pm\alpha}\otimes\Gamma_\tau^{\mp\alpha}\right)\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right) = \frac{b_\alpha(t)}{\lambda}\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right)\left(\Gamma^{\pm\alpha}\otimes\Gamma^{\mp\alpha}\right),$$

where the function $b_\alpha(t)$ depends only on the choice of the reference point λ_0 and the paths C_i, $i=1,2$ and $C(\infty)$.

Proof. According to the conjugation formula (3.5),

$$c_\alpha(\lambda)\left(\Gamma_\tau^{\pm\alpha}\otimes\Gamma_\tau^{\mp\alpha}\right)\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right) = B_\alpha\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right)\left(\Gamma^{\pm\alpha}\otimes\Gamma^{\mp\alpha}\right),$$

where the function B_α is given by the following formula:

$$\begin{aligned} B_\alpha = \quad & \lim_{\epsilon\to 0}\exp\Bigg(\int_{C^{-1}\circ C_1(\epsilon)}\left(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\right)d\xi - (1/2)\int_1^\epsilon\frac{d\xi}{\xi} - \\ & -\frac{t}{4} - \int_{C(\infty)\circ C^{-1}}\left(\left(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\right) - \left(I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi)\right)\right)d\xi\Bigg) = \\ = \quad & e^{-t/4}\lim_{\epsilon\to 0}\exp\left(\int_{C_1(\epsilon)}\left(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\right)d\xi - (1/2)\int_1^\epsilon\frac{d\xi}{\xi}\right)\times \\ & \times\exp\left(-\int_{\lambda_0}^\infty\left(\left(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\right) - \left(I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi)\right)\right)d\xi\right)\times \\ & \times\exp\left(-\int_{\lambda_0}^\lambda\left(I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi)\right)d\xi\right) \end{aligned}$$

Let us discuss each of the terms in the last equality. The first exponential factor is precisely $c_\alpha(\lambda_0)$.

The second exponential factor is some function on t, which depends on the path $C(\infty)$ connecting λ_0 and ∞. The corresponding integral is convergent for the following reason:

$$I_\alpha^{(0)}(\xi,\tau) = \left(1 - S_1\partial_\xi + S_2\partial_\xi^2 \pm \dots\right)I_\alpha^{(0)}(\xi).$$

According to Lemma 3.3, $I_\alpha^{(0)}(\xi) = (1/\xi)P + (1/2)1$. Hence

$$\left(I_\alpha^{(0)}(\xi,\tau), I_\alpha^{(0)}(\xi,\tau)\right) - \left(I_\alpha^{(0)}(\xi), I_\alpha^{(0)}(\xi)\right) = O(\xi^{-2}).$$

The last exponential factor is equal to λ_0/λ. The function b_α is given by the following formula

$$b_\alpha(t) = e^{-t/4}\lim_{R\to\infty}\lim_{\epsilon\to 0}\exp\left(-\int_{u_1+\epsilon}^{R}\Big((I_\alpha^{(0)}(\xi,\tau),I_\alpha^{(0)}(\xi,\tau))\Big)\,d\xi+\int_1^R\frac{d\xi}{\xi}+\int_1^{\epsilon}\frac{d\xi}{2\xi}\right)$$

□

Let us conjugate the vertex operators Γ_τ^δ. If we change the path C by precomposing it with a loop around $\lambda=\infty$, then the corresponding thimble $\alpha(\lambda)$ is transformed into $\alpha(\lambda)+\phi(\lambda)$. Thus using Lemma 3.3, we get the following formula:

$$\mathbf{f}^{\phi}(\lambda) = 2\pi i\sum_{n\geq 0}\frac{\lambda^n}{n!}P(-z)^{-n-1}. \tag{3.15}$$

Introduce the vertex operator

$$\Gamma^\delta = \exp\left\{\left(\mathbf{f}^{\phi/2\pi i}-P(-z)^{-1}\right)^{\widehat{}}\,(\epsilon\partial_x-\log\sqrt{Q})\right\}\exp\left\{(x/\epsilon)\hat{\mathbf{1}}\right\}. \tag{3.16}$$

Lemma 3.5. *The following formula holds:*

$$\left(\Gamma_\tau^{\delta\#}\otimes\Gamma_\tau^{\delta}\right)\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right) = e^{-tx^2/(2\epsilon^2)}\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right)\left(\Gamma^{\delta\#}\otimes\Gamma^{\delta}\right)e^{-tx^2/(2\epsilon^2)}.$$

Proof. Note that $w_\tau = S_\tau\,P(-z)^{-1}$ and $v_\tau = S_\tau\mathbf{1}$. According to (3.5),

$$\widehat{S}_\tau^{-1}\Gamma_\tau^\delta\widehat{S}_\tau = \widehat{S}_\tau^{-1}\exp\left\{\left(\mathbf{f}_\tau^{\phi/2\pi i}-w_\tau\right)^{\widehat{}}\,(\epsilon\partial_x-\log\sqrt{Q})\right\}\exp\left\{(x/\epsilon)\hat{v}_\tau\right\}\hat{S}_\tau =$$
$$\exp\left\{\left(\mathbf{f}^{\phi/2\pi i}-P(-z)^{-1}\right)^{\widehat{}}\,(\epsilon\partial_x-\log\sqrt{Q})\right\}\exp\left\{(x/\epsilon)\hat{\mathbf{1}}\right\}e^{-W_\tau(\mathbf{1},\mathbf{1})\frac{x^2}{2\epsilon^2}}$$

i.e., $\widehat{S}_\tau^{-1}\Gamma_\tau^\delta\widehat{S}_\tau = \Gamma^\delta e^{-tx^2/(2\epsilon^2)}$. Similarly, $\widehat{S}_\tau^{-1}\Gamma_\tau^{\delta\#}\widehat{S}_\tau = e^{-tx^2/(2\epsilon^2)}\Gamma^{\delta\#}$. □

3.4. **Proof of Corollary 1.2.** Let $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{D}}$ be the 1-forms respectively (1.7) and

$$b_\alpha\left(\Gamma^{\delta\#}\otimes\Gamma^\delta\right)\left(\Gamma^\alpha\otimes\Gamma^{-\alpha}-\Gamma^{-\alpha}\otimes\Gamma^\alpha\right)\left(\mathcal{D}(\mathbf{q}')\mathcal{D}(\mathbf{q}'')\right)\frac{d\lambda}{\lambda}$$

According to Lemma 3.4 and Lemma 3.5, and formula (3.2) for the descendants in terms of the ancestors,

$$e^{2\mathrm{F}^{(1)}(\tau)}\omega_{\mathcal{A}} = e^{-tx^2/(2\epsilon^2)}\left(\widehat{S}_\tau\otimes\widehat{S}_\tau\right)\omega_{\mathcal{D}}e^{-tx^2/(2\epsilon^2)}.$$

Using formula (3.3) for the action of $\widehat{S}_\tau$ on the Fock space, we find that up to factors independent of λ the 1-forms $\omega_{\mathcal{D}}(\mathbf{q}',\mathbf{q}'';\lambda)$ and $\omega_{\mathcal{A}}([S_\tau\mathbf{q}']_+,[S_\tau\mathbf{q}'']_+;\lambda)$ coincide. On the other hand, since S_τ is a symplectic transformation of $\mathcal{H}$ and $w_\tau = S_\tau(P(-z)^{-1})$, we get

$$\begin{aligned}\hat{w}_\tau([S_\tau\mathbf{q}']_+)-\hat{w}_\tau([S_\tau\mathbf{q}'']_+) &= \epsilon^{-1}\Omega\left([S_\tau\mathbf{q}']_+-[S_\tau\mathbf{q}'']_+,w_\tau\right) = \\ =\epsilon^{-1}\Omega\left(S_\tau\mathbf{q}'-S_\tau\mathbf{q}'',w_\tau\right) &= \epsilon^{-1}\Omega\left(\mathbf{q}'-\mathbf{q}'',P(-z)^{-1}\right) = (q_{0,0}''-q_{0,0}')/\epsilon.\end{aligned}$$

Thus according to Theorem 1.1, the total descendant potential satisfies the following HQE: for each integer m the 1-form

$$\left(\Gamma^{\delta\#}\otimes\Gamma^\delta\right)\left(\Gamma^\alpha\otimes\Gamma^{-\alpha}-\Gamma^{-\alpha}\otimes\Gamma^\alpha\right)\left(\mathcal{D}(\mathbf{q}')\mathcal{D}(\mathbf{q}'')\right)\frac{d\lambda}{\lambda},$$

when computed at $q'_{0,0} - q''_{0,0} = m\epsilon$ is regular in λ. These are precisely the HQE of the Extended Toda Hierarchy which were introduced in [M]. According to Theorem 1.1 in [M], the total descendant potential of $\mathbb{C}P^1$ is a tau-function of the Extended Toda Hierarchy. $\square$

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